

Energy scaling and branched microstructures in a model for shape-memory alloys with $\text{SO}(2)$ invariance

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Domain branching near the boundary appears in many singularly-perturbed models for microstructure in materials and was first demonstrated mathematically by Kohn and Müller for a scalar problem modeling the elastic behavior of shape-memory alloys. We study here a model for shape-memory alloys based on the full vectorial problem of nonlinear elasticity, including invariance under rotations, in the case of two wells in two dimensions. We show that, for two wells with two rank-one connections, the energy scales proportional to the power $2/3$ of the surface energy, in agreement with the scalar model. In a case where only one rank-one connection is present, we show that the energy exhibits a different behavior, proportional to the power $4/5$ of the surface energy. This lower energy is achieved by a suitable interaction of the two components of the deformations and hence cannot be reproduced by the scalar model. Both scalings are proven by explicit constructions and matching lower bounds.

1 Introduction

The peculiar elastic properties of shape-memory alloys have attracted a large interest in the mechanics literature, as discussed for example in [Bha03, OW99]; starting with the works of Ball and James [BJ87, BJ92] also in the vectorial calculus of variations, see [CK88, Dac89, Mül99] and the references therein. The typical model, based on nonlinear elasticity, takes the form

$$\int_{\Omega} W(Du) dx \tag{1.1}$$

where $u : \Omega \rightarrow \mathbb{R}^n$ is the elastic deformation. The energy density $W : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is minimized by several copies of the set of proper rotations $\text{SO}(n)$, which correspond to the several phases of the material. The macroscopic material behavior can be analyzed using the theory of relaxation, which – under appropriate continuity and growth conditions – leads to the study of the quasiconvex envelope of W [Dac89, Mül99]. Partial results in this direction have been obtained for example in [BJ92, Šve93, Dol03, DK03,

CDK07, CD14]. The theory of relaxation does not, however, make any prediction on the length scale of the microstructure, as the functional (1.1) is scale invariant.

A more precise analysis of the microstructure is possible if one includes a singular perturbation, typically of the form $\varepsilon|D^2u|(\Omega)$ (in the sense of the total variation of the gradient of the BV function Du) or $\varepsilon^2\|D^2u\|_{L^2}^2$, where ε represents a (scaled) surface energy. A scalar simplification of this type of model was proposed by Kohn and Müller in 1992 [KM92, KM94]. They have shown that the minimum energy scales as $\varepsilon^{2/3}$, and that this scaling is achieved by a self-similar microstructure which refines close to the boundary. Finer results were later obtained in [Con00, Con06, Zwi12], including in particular the statement of asymptotic self-similarity for the minimizers of the Kohn-Müller functional. Their approach was extended to many other physical problems, including for example micromagnetism [CK98, CKO99, Vie09], magnetic structures in superconductors [CKO04, CCKO08], dislocation structures in plasticity [CO05], coarsening in thin film growth [CO08].

We consider in this work the full vectorial problem of nonlinear elasticity with surface energy, namely, the functional

$$\int_{\Omega} W(Du) dx + \varepsilon|D^2u|(\Omega). \quad (1.2)$$

Here, $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents the elastic deformation, $W : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ the energy density which is minimized on a set $K \subset \mathbb{R}^{n \times n}$ and grows quadratically; for definiteness we shall focus on

$$W(F) = \text{dist}^2(F, K) = \inf\{|F - G|^2 : G \in K\} \quad (1.3)$$

although all results hold for any energy with the same set of minimizers and the same growth. The small positive parameter ε in (1.2) represents the surface energy, and the admissible deformations are those $u \in W^{1,2}(\Omega; \mathbb{R}^n)$ with $Du \in BV(\Omega; \mathbb{R}^{n \times n})$ which obey specific boundary conditions. By $|D^2u|(\Omega)$ we denote the total variation of the measure D^2u over the set Ω . The set $K \subset \mathbb{R}^{n \times n}$ of energy-minimizing deformation gradients, as used in (1.3), depends on the crystallography of the phase transformation considered.

In this paper we focus on the two-dimensional case, $n = 2$, and study two different crystallographic settings. The first situation we address, denoted case K_1 , is

$$A_1 = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad K_1 = \text{SO}(2)A_1 \cup \text{SO}(2)B_1 \quad (1.4)$$

for some $\alpha > 0$. These matrices have the same determinant and are representative of volume-preserving phase transformations in which the two wells have two rank-one connections, in the sense that $\det(A_1 - QB_1) = 0$ has two

solutions $Q \in \text{SO}(2)$. This is the vectorial situation that was represented by the scalar model of Kohn and Müller; indeed, one can obtain a model similar to the one they used if K_1 is reduced to the two matrices $\{A_1, B_1\}$, see [Zwi12]. The two components of u are in this case decoupled, only the first one gives a nontrivial energy contribution.

For this problem we show that the Kohn-Müller scaling proportional to $\varepsilon^{2/3}$ holds also for the vectorial case, if the aspect ratio of the domain is close to 1 (Theorem 1.1). The presence of a second rank-one connection becomes, however, important in the case of long and thin domains, in which microstructure is generated along vertical oscillations instead of horizontal ones, as we discuss after Theorem 1.1 and in Section 2.2 below.

The second situation we consider, denoted case K_2 , is

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \alpha \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 + \alpha \end{pmatrix}, \quad K_2 = \text{SO}(2)A_2 \cup \text{SO}(2)B_2 \quad (1.5)$$

for some $\alpha \in (0, 1/2)$. These matrices have a different determinant and are representative of phase transformations with one rank-one connection, in the sense that $\det(A_2 - QB_2) = 0$ has a unique solution $Q \in \text{SO}(2)$. In this case the invariance under rotations and the vectorial nature of the deformation can interact to reduce the energy cost of domain branching. The resulting scaling, proportional to $\varepsilon^{4/5}$, differs from the scalar problem. This difference can be understood as a consequence of the degeneracy of the rank-one connection (in the same sense as for double roots of polynomials), details will become apparent in the construction discussed in Section 2.1, see in particular Lemma 2.1.

If no rank-one connection is present, then the energy is much larger, and typically no microstructure is formed.

Before stating the main results we introduce the key parameters and the set of admissible functions. We focus on a rectangle, and assume that

$$\Omega = (0, L) \times (0, H), \quad H, L > 0, \quad \alpha \in (0, 1/2), \quad \varepsilon > 0. \quad (1.6)$$

We shall impose Dirichlet boundary conditions corresponding to the average of A_j and B_j , which with the present choices is in both cases the identity, $\frac{1}{2}(A_j + B_j) = \text{Id}$. Precisely, the admissible deformations are in the set

$$\mathcal{M} = \{u \in W^{1,2}(\Omega; \mathbb{R}^2) : Du \in BV(\Omega; \mathbb{R}^{2 \times 2}), u(x) = x \ \forall x \in \partial\Omega\}. \quad (1.7)$$

For $j \in \{1, 2\}$ we are interested in the scaling of the minimum of

$$E_j^\varepsilon[u, \Omega] = \int_\Omega \text{dist}^2(Du, K_j) dx + \varepsilon |D^2 u|(\Omega) \quad (1.8)$$

over all $u \in \mathcal{M}$. Existence of minimizers follows immediately from the direct method of the calculus of variations and will not be further addressed here.

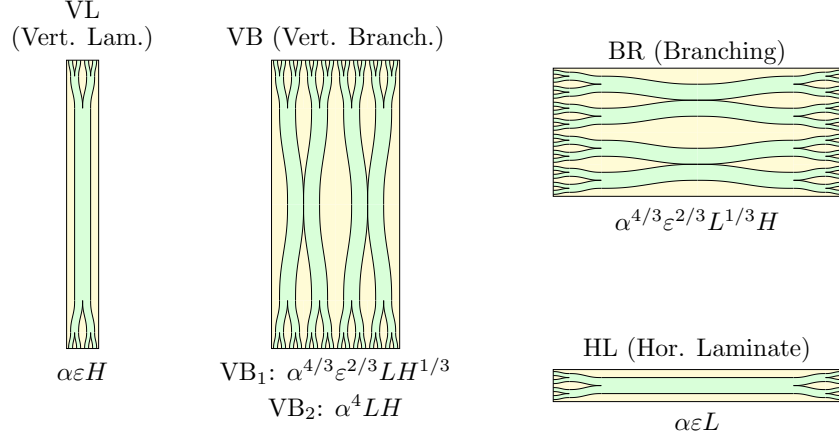


Figure 1: Sketches of the deformation u in the different phases appearing in Theorem 1.1.

In the following c will denote a generic positive constant, which may change its value from line to line but in particular does not depend on the four parameters α , ε , H and L . We shall denote by $E_j^\varepsilon[v, \omega]$ the integral of the same expression over a subset $\omega \subset \Omega$.

Theorem 1.1. *There is $c > 0$ such that, under the assumptions (1.4), (1.6), (1.7) and (1.8), one has*

$$\frac{1}{c}f(\alpha, \varepsilon, L, H) \leq \min_{u \in \mathcal{M}} E_1^\varepsilon[u, \Omega] \leq cf(\alpha, \varepsilon, L, H)$$

where

$$f(\alpha, \varepsilon, L, H) = \min \left\{ \begin{aligned} &\alpha^{4/3} \varepsilon^{2/3} L^{1/3} H + \alpha \varepsilon L, \\ &\alpha^{4/3} \varepsilon^{2/3} L H^{1/3} + \alpha^4 L H + \alpha \varepsilon H, \\ &\alpha^2 L H \end{aligned} \right\}. \quad (1.9)$$

The constant c does not depend on α , ε , L and H .

Proof. Follows from Lemma 2.6 and Lemma 3.5. \square

Theorem 1.1 shows that, depending on the values of the four parameters, different types of microstructure give the optimal energy scaling. We sketch in Figure 1 the different microstructures (using the constructions from the upper bound) and in Figure 2 the corresponding phase diagram. We now briefly discuss the different phases.

- A If the grain is very small then the boundary data and the regularization term dominate the picture, the transformation to martensite is not

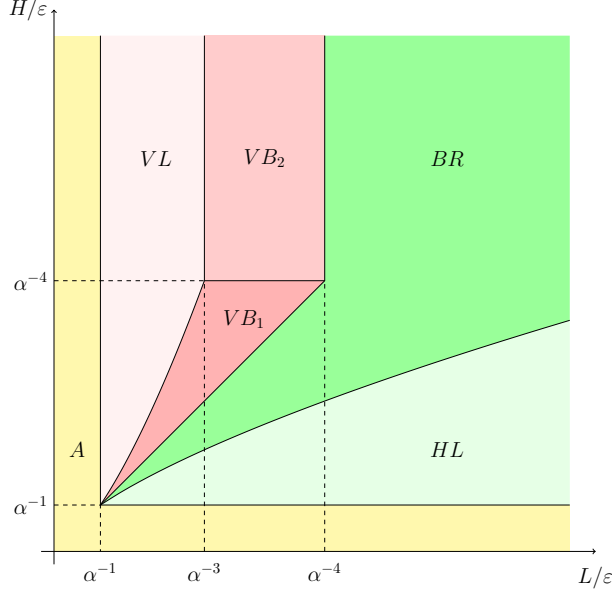


Figure 2: Schematic phase diagram in case K_1 , in the $L/\varepsilon, H/\varepsilon$ plane.

energetically convenient. The energy scales as $\alpha^2 LH$, which is easily attained by the deformation $u(x) = x$. We call this austenitic (A) phase. From (1.9) one easily sees that this phase is the optimal one if $L/\varepsilon < \alpha^{-1}$ or $H/\varepsilon < \alpha^{-1}$.

BR In the branching regime (BR) the deformation gradient Du oscillates between a value close to A_1 and a value close to B_1 in the interior of the sample; the boundaries are approximately horizontal and the oscillations become finer approaching the left and right boundary. The total energy, if L and H are sufficiently large compared to ε , is proportional to $\alpha^{4/3} \varepsilon^{2/3} L^{1/3} H$, as predicted by Kohn and Müller.

HL Horizontal laminate. If the sample is long and thin, then it is convenient to have a single oscillation in most of the sample, which then branches only in small regions close to the left and right boundary. The dominant energy contribution originates then from the single long interface and is proportional to $\alpha \varepsilon L$.

VB Vertical branching: if H is larger than L , then it may be more convenient to use the second rank-one connection, and have a vertical branching, qualitatively similar to a 90-degrees rotation of regime BR. The rotation, since we are dealing with a fully nonlinear model, does not bring to the matrices A_1, B_1 to the matrices $\text{Id} \pm \alpha e_1 \otimes e_2$, but instead leaves a second-order perturbation, which generates an additional energy cost proportional to $\alpha^4 LH$ (see discussion in Lemma 2.6

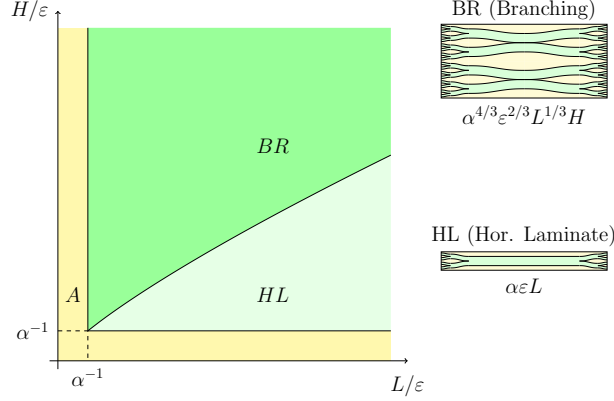


Figure 3: Schematic phase diagram in case K_2 , in the L/ε , H/ε plane.

for details). Therefore this regime can be subdivided into two parts: for small H , the $\alpha^{4/3}\varepsilon^{2/3}LH^{1/3}$ term dominates (regime VB₁), for large H instead the α^4LH term dominates (regime VB₂).

VL As in the case of phase HL if L is very small then a long vertical interface will dominate.

The boundaries between the different regions can be easily obtained from (1.9), and are sketched in Figure 2.

In the case of a single rank-one connection the energy scaling is different, and the phase diagram simpler, see Figure 3.

Theorem 1.2. *There is $c > 0$ such that, under the assumptions (1.5), (1.6), (1.7) and (1.8), one has*

$$\frac{1}{c}g(\alpha, \varepsilon, L, H) \leq \min_{u \in \mathcal{M}} E_2^\varepsilon[u, \Omega] \leq cg(\alpha, \varepsilon, L, H)$$

where

$$g(\alpha, \varepsilon, L, H) = \min \left\{ \alpha^{6/5} \varepsilon^{4/5} L^{1/5} H + \alpha \varepsilon L, \alpha^2 L H \right\}.$$

The constants c does not depend on α , ε , H and L .

Proof. Follows from Lemma 2.3 and Lemma 3.6. \square

The situation in which the dependence on α is not analyzed is substantially simpler, in particular for the lower bounds. Indeed, a result similar to Theorem 1.1 without dependence on α and with $H = L$ was first proven in [Cha07] and announced in [Con07]. These results are discussed in the PhD thesis [Cha13] and, in the simplified case $\alpha \sim 1$ and $H = L$, were presented in [CC14]. A version of Theorem 1.1 with a geometrically linear model and without the explicit dependence on H and L was obtained in [Die13]. The upper bounds are proven in Section 2, the lower bounds in Section 3.

2 Upper bounds

The upper bounds are proven by explicit constructions. Whereas the basic construction in the proof of Theorem 1.1 is a direct generalization of the one by Kohn and Müller, the different scaling in Theorem 1.2 arises from a somewhat different construction, which exploits the vectorial nature of the problem and the invariance under rotations of the energy density. Therefore we start with a discussion of the latter, and postpone the proof of Theorem 1.1 to Section 2.2 below. In what follows c denotes a positive constant which does not depend on α , ε , L and H and may be different from line to line. In this section we write (x, y) for generic points in \mathbb{R}^2 .

2.1 Proof of the upper bound in Theorem 1.2

The key idea is to use a number of period-doubling steps which join fine-scale oscillations close to the boundary with coarse-scale oscillations in the interior of the sample. The difference between the two cases considered, and the main novelty in the present construction, resides in the treatment of the period-doubling step, which we now present.

In order to fix the boundary conditions on the internal boundaries, we fix a continuous 1-periodic function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\theta(0) = 0$, $\theta' = 1$ on $(-1/4, 1/4)$, $\theta' = -1$ on $(1/4, 3/4)$ (equivalently, $\theta(t) = \text{dist}(t + \frac{1}{4}, \mathbb{Z}) - \frac{1}{4}$) and scale it according to $\theta_h(t) = h\theta(t/h)$. Then θ_h is h -periodic and has derivative ± 1 almost everywhere. We observe that the function

$$w_h(x, y) = \begin{pmatrix} x \\ y + \alpha\theta_h(y) \end{pmatrix}$$

fulfills $Dw_h \in \{A_2, B_2\}$ almost everywhere. Further, on all lines $y \in h\mathbb{Z}$ we have $w_h(x, y) = (x, y)$, corresponding to the prescribed boundary conditions. The construction will be based on using $w_h(x, y)$ for selected values of x , with different values of h , larger in the center of the domain and smaller close to $x = 0$ and $x = L$. The key ingredient in the proof is the following period-doubling construction which permits to join different values of h with small energetic cost.

Lemma 2.1. *Let $0 < h \leq \ell$, $\omega = (x_0, x_0 + \ell) \times (y_0, y_0 + h)$, $\alpha, \varepsilon \in (0, 1)$. Then there is a function $v : \omega \rightarrow \mathbb{R}^2$ such that*

$$E_2^\varepsilon[v, \omega] \leq c \left(\alpha^2 \frac{h^5}{\ell^3} + \alpha\varepsilon\ell \right)$$

with the boundary conditions $v(x, y) = (x, y)$ for $y \in \{y_0, y_0 + h\}$, $v(x_0, y) = (x_0, y + \alpha\theta_{h/2}(y - y_0))$, $v(x_0 + \ell, y) = (x_0 + \ell, y + \alpha\theta_h(y - y_0))$ and $\|Dv - \text{Id}\|_{L^\infty} \leq c\alpha$.

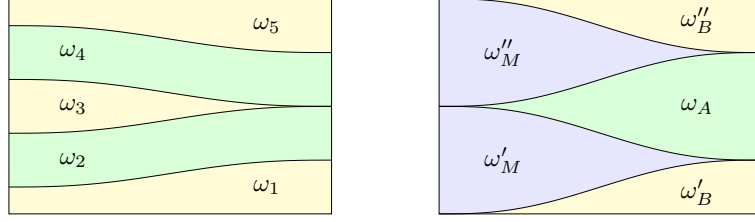


Figure 4: Left panel: subdivision of the domain ω used in Lemma 2.1. Right panel: subdivision of the domain used in Lemma 2.2.

Proof. We assume without loss of generality $x_0 = y_0 = 0$. We fix a smooth interpolation function $\gamma : [0, 1] \rightarrow [0, 1]$ with $\gamma(0) = 0$, $\gamma(1) = 1$ and the first two derivatives vanishing at both points, say $\gamma(x) = 10x^3 - 15x^4 + 6x^5$. We subdivide ω into the five sets

$$\begin{aligned}\omega_1 &= \left\{ (x, y) \in \omega : 0 < y < \frac{h}{8} + \frac{h}{8}\gamma\left(\frac{x}{\ell}\right) \right\}, \\ \omega_2 &= \left\{ (x, y) \in \omega : \frac{h}{8} + \frac{h}{8}\gamma\left(\frac{x}{\ell}\right) < y < \frac{3h}{8} + \frac{h}{8}\gamma\left(\frac{x}{\ell}\right) \right\}, \\ \omega_3 &= \left\{ (x, y) \in \omega : \frac{3h}{8} + \frac{h}{8}\gamma\left(\frac{x}{\ell}\right) < y < \frac{5h}{8} - \frac{h}{8}\gamma\left(\frac{x}{\ell}\right) \right\}, \\ \omega_4 &= \left\{ (x, y) \in \omega : \frac{5h}{8} - \frac{h}{8}\gamma\left(\frac{x}{\ell}\right) < y < \frac{7h}{8} - \frac{h}{8}\gamma\left(\frac{x}{\ell}\right) \right\}, \\ \omega_5 &= \left\{ (x, y) \in \omega : \frac{7h}{8} - \frac{h}{8}\gamma\left(\frac{x}{\ell}\right) < y < h \right\},\end{aligned}$$

see Figure 4. In the domains ω_1 , ω_3 and ω_5 we shall construct v so that Dv is close to $B_2 = \text{diag}(1, 1 + \alpha)$, in the other two so that it is close to $A_2 = \text{diag}(1, 1 - \alpha)$. Since the boundaries are curved, there will necessarily be deviations from A_2 and B_2 , and in particular the x -derivative will not be identically e_1 .

The basic idea is to set $v_2(\cdot, 0) = 0$, according to the boundary conditions, and then to construct $v_2(x, y)$ by integrating $\partial_2 v_2(x, \cdot)$ from 0 to y , with $\partial_2 v_2 = 1 + \alpha$ in ω_1 , ω_3 , ω_5 , and $\partial_2 v_2 = 1 - \alpha$ on the other two sets. This leads to

$$v_2(x, y) = \begin{cases} (1 + \alpha)y & \text{on } \omega_1, \\ (1 - \alpha)y + \alpha h\left(\frac{1}{4} + \frac{1}{4}\gamma\left(\frac{x}{\ell}\right)\right) & \text{on } \omega_2, \\ (1 + \alpha)y - \frac{\alpha h}{2} & \text{on } \omega_3, \\ (1 - \alpha)y + \alpha h\left(\frac{3}{4} - \frac{1}{4}\gamma\left(\frac{x}{\ell}\right)\right) & \text{on } \omega_4, \\ (1 + \alpha)y - \alpha h & \text{on } \omega_5. \end{cases}$$

Computing $\partial_1 v_2$ we easily see that it is of order $\alpha h/\ell$ in ω_2 and ω_4 . In

particular, if we would simply set $v_1(x, y) = x$ we would obtain

$$|Dv - A_2| \sim \frac{\alpha h}{\ell} \text{ in } \omega_2 \text{ and } \omega_4 \quad (2.1)$$

and therefore an energy at least of order $\alpha^2 h^3 / \ell$ in ω .

It is however possible, via a more careful construction of v_1 , to reduce the energy further. The idea is to use rotations, in a way similar to what was done in the study of folding pattern in blistered thin films [BCDM00, BCDM02]. To understand the key idea it is helpful to consider first the linearized model, in which $\text{dist}(Dv, \text{SO}(2)A_2)$ is replaced by $|(Dv + Dv^T)/2 - A_2|$. The easier choice for v_1 , namely, $v_1(x, y) = x$, would make the 11 entry of this matrix vanish. The other natural option, namely, to make the off-diagonal entries (which are equal) vanish, leads to the possibility of defining v_1 by solving $\partial_2 v_1 + \partial_1 v_2 = 0$. By the same estimate as in (2.1), the term $\partial_1 v_2$ is of order $\alpha h / \ell$, hence integrating in y we would obtain a component v_1 of order $\alpha h^2 / \ell$. The diagonal entry in the gradient $\partial_1 v_1$ would then be of order $\alpha (h / \ell)^2$. The additional factor h / ℓ in $\partial_1 v_1$ with respect to $\partial_2 v_1$ arises from the fact that we took an integral in the y direction and a derivative in the x -direction. On most of our domain h will be substantially smaller than ℓ , therefore this construction leads to a substantial reduction in energy.

Since we are dealing with a geometrically nonlinear model there will be an additional error term arising from the linearization, of order $(\partial_1 v_2)^2 \sim \alpha^2 (h / \ell)^2$. We also need to take care of the fact that we are not linearizing around the identity, but instead around the matrix A_2 , therefore we solve $\partial_2 v_1 + (1 - \alpha) \partial_1 v_2 = 0$ instead of $\partial_2 v_1 + \partial_1 v_2 = 0$, see (2.2) below.

A detailed computation based on this idea motivates the definition

$$v_1(x, y) = \begin{cases} x & \text{on } \omega_1, \\ x + \alpha(1 - \alpha) \frac{h}{4\ell} \gamma' \left(\frac{x}{\ell} \right) \left(\frac{h}{8} + \frac{h}{8} \gamma \left(\frac{x}{\ell} \right) - y \right) & \text{on } \omega_2, \\ x - \alpha(1 - \alpha) \gamma' \left(\frac{x}{\ell} \right) \frac{h^2}{16\ell} & \text{on } \omega_3, \\ x - \alpha(1 - \alpha) \frac{h}{4\ell} \gamma' \left(\frac{x}{\ell} \right) \left(\frac{7h}{8} - \frac{h}{8\ell} \gamma \left(\frac{x}{\ell} \right) - y \right) & \text{on } \omega_4, \\ x & \text{on } \omega_5. \end{cases}$$

This concludes the definition of v , in the rest of the proof we estimate its energy. First we easily check that v is continuous. By construction, $Dv = B_2 \in K_2$ in the sets ω_1 and ω_5 . A short computation shows that $|Dv - B_2| = |\partial_1 v_1 - 1| \leq c \alpha h^2 / \ell^2$ in ω_3 . In order to estimate the energy in ω_2 and ω_4 we compute

$$\begin{aligned} \min_{Q \in \text{SO}(2)} |Dv - QA_2| &\leq \min_{\varphi \in \mathbb{R}} |\partial_1 v_1 - \cos \varphi| + |\partial_2 v_2 - (1 - \alpha) \cos \varphi| \\ &\quad + |\partial_1 v_2 - \sin \varphi| + |\partial_2 v_1 + (1 - \alpha) \sin \varphi| \\ &\leq |\partial_1 v_1 - 1| + |\partial_2 v_2 - (1 - \alpha)| \\ &\quad + |\partial_2 v_1 + (1 - \alpha) \partial_1 v_2| + c |\partial_1 v_2|^2, \end{aligned} \quad (2.2)$$

where we have chosen φ such that $\sin \varphi = \partial_1 v_2$ if $\partial_1 v_2 \in [-1, 1]$, $\varphi = 0$ otherwise. Inserting the definition of v one sees that the second and third term vanish, the remainder can be estimated by

$$\begin{aligned} \text{dist}(Dv, K) &\leq c\alpha \frac{h^2}{\ell^2} \|\gamma''\|_{L^\infty} + c\alpha \frac{h^2}{\ell^2} \|\gamma'\|_{L^\infty}^2 + c \left(\alpha \frac{h}{\ell} \|\gamma'\|_{L^\infty} \right)^2 \\ &\leq c\alpha \frac{h^2}{\ell^2} \quad \text{in } \omega_2 \cup \omega_4 \end{aligned}$$

and therefore

$$\int_{\omega_2 \cup \omega_4} \text{dist}^2(Dv(x, y), K_2) dx dy \leq c\alpha^2 \frac{h^5}{\ell^3}.$$

A simple computation shows that $|Dv - \text{Id}| \leq c\alpha$ on ω .

We finally turn to the interfacial part of the energy. In each of the ω_i we see by inspection that $|D^2 v| \leq c\alpha h/\ell^2$ (we use repeatedly that $h \leq \ell$ to simplify expressions). The jump contributions can be estimated by $2\|Dv - \text{Id}\|_{L^\infty}$ times the length of the jump set; since γ is monotone this is bounded by $4(h + \ell)$. We conclude that

$$\varepsilon |D^2 v|(\omega) \leq c\varepsilon \left(\alpha(\ell + h) + \alpha h \ell \frac{h}{\ell^2} \right) \leq c\varepsilon \alpha \ell.$$

This concludes the proof. \square

We now turn to the boundary layer. One could use a linear interpolation as in [CC14]. For variety we present here an alternative construction, which is based on a variant of the construction of Lemma 2.1 and does not use gradients far from A , B and Id .

Lemma 2.2. *Let $0 < h \leq \ell$, $\omega = (x_0, x_0 + \ell) \times (y_0, y_0 + h)$, $\alpha \in (0, 1)$, $\varepsilon > 0$. Then there is a function $v : \omega \rightarrow \mathbb{R}^2$ such that*

$$E_2^\varepsilon[v, \omega] \leq c(\alpha^2 h \ell + \alpha \varepsilon \ell)$$

with the boundary conditions $v(x, y) = (x, y)$ for $y \in \{y_0, y_0 + h\}$ or $x = x_0$, $v(x_0 + \ell, y) = (x_0 + \ell, y + \alpha \theta_h(y - y_0))$ and $\|Dv - \text{Id}\|_{L^\infty} \leq c\alpha$.

Proof. The construction is similar to the one in Lemma 2.1. Again it suffices

to treat the case $x_0 = y_0 = 0$. With γ as in Lemma 2.1 we set

$$\begin{aligned}\omega'_B &= \left\{ x \in (0, \ell), \ 0 < y < \frac{h}{4}\gamma\left(\frac{x}{\ell}\right) \right\}, \\ \omega'_M &= \left\{ x \in (0, \ell), \ \frac{h}{4}\gamma\left(\frac{x}{\ell}\right) < y < \frac{h}{2} - \frac{h}{4}\gamma\left(\frac{x}{\ell}\right) \right\}, \\ \omega_A &= \left\{ x \in (0, \ell), \ \frac{h}{2} - \frac{h}{4}\gamma\left(\frac{x}{\ell}\right) < y < \frac{h}{2} + \frac{h}{4}\gamma\left(\frac{x}{\ell}\right) \right\}, \\ \omega''_M &= \left\{ x \in (0, \ell), \ \frac{h}{2} + \frac{h}{4}\gamma\left(\frac{x}{\ell}\right) < y < h - \frac{h}{4}\gamma\left(\frac{x}{\ell}\right) \right\}, \\ \omega''_B &= \left\{ x \in (0, \ell), \ h - \frac{h}{4}\gamma\left(\frac{x}{\ell}\right) < y < h \right\},\end{aligned}$$

see Figure 4. We set $v_1(x, y) = x$ and

$$v_2(x, y) = \begin{cases} y + \alpha y & \text{on } \omega'_B, \\ y + \frac{1}{4}\alpha h \gamma\left(\frac{x}{\ell}\right) & \text{on } \omega'_M, \\ y + \alpha\left(\frac{h}{2} - y\right) & \text{on } \omega_A, \\ y - \frac{1}{4}\alpha h \gamma\left(\frac{x}{\ell}\right) & \text{on } \omega''_M, \\ y + \alpha(y - h) & \text{on } \omega''_B, \end{cases}$$

One then checks that v_2 is continuous, $|D^2 v_2| \leq c\alpha h/\ell^2$ inside each of the five sets, $Dv = B_2$ on $\omega'_B \cup \omega''_B$, $Dv = A_2$ in ω_A , and $|Dv - \text{Id}| \leq \alpha h \|\gamma'\|_{L^\infty}/\ell$ in $\omega'_M \cup \omega''_M$. Therefore $\|Dv - \text{Id}\|_{L^\infty} \leq c\alpha$ and, treating the jump terms as above,

$$E_2^\varepsilon[u, (0, \ell) \times (0, h)] \leq c\alpha^2 h \ell + c\varepsilon \alpha \ell.$$

□

We are now ready to present the full construction.

Lemma 2.3. *Under the assumptions of Theorem 1.2 there is a deformation $u \in \mathcal{M}$ such that*

$$E_2^\varepsilon[u, \Omega] \leq c \min \left\{ \alpha^{6/5} \varepsilon^{4/5} L^{1/5} H + \alpha \varepsilon L, \alpha^2 L H \right\}.$$

Proof. Inserting $u(x) = x$ we easily obtain a construction with energy $\alpha^2 L H$. Therefore we only need to consider the case in which the first term in the minimum is smaller than the second. Then necessarily $\varepsilon^{4/5} \alpha^{6/5} L^{1/5} H \leq \alpha^2 L H$, which is equivalent to

$$\varepsilon \leq \alpha L. \tag{2.3}$$

We now start the construction. We will focus on $\Omega' = (0, L/2) \times (0, H)$, the other part is analogous (with minor sign changes in the formulas). The

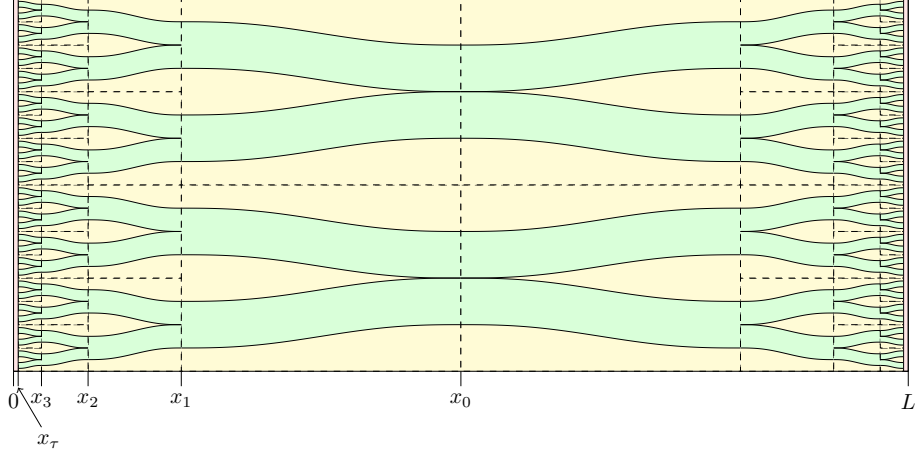


Figure 5: Global pattern used in Lemma 2.3 for the construction of the upper bound of Theorem 1.2.

basic idea in the construction is to have coarse folds in the central part, which then refine in a self-similar way approaching to the boundary, up to a point where a different construction for the boundary layer is needed, see Figure 5.

Let $N \in \mathbb{N} \setminus \{0\}$ be the number of oscillations in the center, $\theta \in (1/4, 1/2)$ a geometric factor, both will be chosen below. We cut the domain in vertical stripes, so that in each of them, say $(x_{i+1}, x_i) \times (0, H)$, the construction will be h_i -periodic in the y direction. We define

$$x_i = \frac{L}{2}\theta^i \quad \text{and} \quad h_i = \frac{H}{2^i N}$$

and denote the width of the i -th stripe by $\ell_i = \theta^i(1 - \theta)\frac{L}{2}$.

Since the construction in Lemma 2.1 requires $h_i \leq \ell_i$, we need to stop this procedure at a finite τ , defined as the largest integer i with $h_i \leq \ell_i$. Further, we need to choose N such that $H/N \leq (1 - \theta)L/2$, to be able to start.

For $i \in [0, \tau] \cap \mathbb{N}$ and $y \in (0, L)$ we set

$$v(x_i, y) = \left(\begin{array}{c} x_i \\ y + \alpha \theta_{h_i}(y) \end{array} \right). \quad (2.4)$$

This permits to treat each refinement step separately. In each stripe $(x_{i+1}, x_i) \times (0, H)$, $i = 0, \dots, \tau - 1$, we use $N2^i$ times the construction of Lemma 2.1. In the set $(0, x_\tau) \times (0, L)$ we use the same procedure with Lemma 2.2. The result is continuous thanks to (2.4).

It remains to estimate the energy. This is given by the sum over the stripes of $N2^i$ times the energy in each rectangle plus the boundary energy,

which on the boundary of each rectangle is controlled by $\varepsilon \|Dv - \text{Id}\|_{L^\infty}(h_i + \ell_i) \leq c\alpha\varepsilon\ell_i$. We obtain

$$E_2^\varepsilon[u, \Omega'] \leq c \sum_{i=0}^{\tau-1} 2^i N \left(\alpha^2 \frac{h_i^5}{\ell_i^3} + \alpha\varepsilon\ell_i \right) + cN2^\tau (\alpha^2 h_\tau \ell_\tau + \alpha\varepsilon\ell_\tau).$$

By definition of τ we have $h_\tau \leq \ell_\tau \leq 2h_\tau$, therefore the boundary term is comparable to the $i = \tau$ term in the series and can be included in the series. Using the definitions of h_i and ℓ_i the estimate becomes

$$E_2^\varepsilon[u, \Omega'] \leq c \sum_{i=0}^{\tau} \left(\frac{\alpha^2 H^5}{2^{4i} N^4 \theta^{3i} L^3 (1-\theta)^3} + \alpha\varepsilon 2^i N \theta^i (1-\theta)L \right).$$

If $\theta \in (2^{-4/3}, 2^{-1})$ both $\sum_i 1/(2^4 \theta^3)^i$ and $\sum_i (2\theta)^i$ are converging geometric series. We choose $\theta = 2^{-5/4}$ and obtain

$$E_2^\varepsilon[u, \Omega'] \leq c \left(\frac{\alpha^2 H^5}{N^4 L^3} + \alpha\varepsilon N L \right).$$

We choose N so that the sum is smallest, subject to the constraints $N \geq 1$, $N \in \mathbb{N}$, $N \geq 4H/L$. Precisely, we set $N = \lceil \alpha^{1/5} H / (\varepsilon^{1/5} L^{4/5}) + 4H/L \rceil$, where $\lceil t \rceil = \min\{z \in \mathbb{Z} : z \geq t\}$, and conclude

$$E_2^\varepsilon[u, \Omega'] \leq c\alpha^{6/5} \varepsilon^{4/5} H L^{1/5} + c\alpha\varepsilon L + c\alpha\varepsilon H.$$

Including the set $\Omega \setminus \Omega'$ only doubles the energy. By (2.3) we see that the term $\alpha\varepsilon H$ can be absorbed into the first one, and the proof is concluded. \square

2.2 Proof of the upper bound in Theorem 1.1

The proof is similar to the one of Theorem 1.2, but the construction of the period-doubling step in Lemma 2.1 becomes simpler. We only discuss the differences. The presence of two rank-one connections however leads in this case to the presence of two competing constructions, which are essentially 90-degrees rotated copies of each other. The relation is discussed in the proof of Lemma 2.6 below.

Lemma 2.4. *Let $0 < h \leq \ell$, $\omega = (x_0, x_0 + \ell) \times (y_0, y_0 + h)$, $\alpha \in (0, 1)$, $\varepsilon > 0$. Then there is a function $v : \omega \rightarrow \mathbb{R}^2$ such that*

$$E_1^\varepsilon[v, \omega] \leq c \left(\alpha^2 \frac{h^3}{\ell} + \alpha\varepsilon\ell \right)$$

with the boundary conditions $v(x, y) = (x, y)$ for $y \in \{y_0, y_0 + h\}$, $v(x_0, y) = (x_0 \alpha \theta_{h/2}(y - y_0), y)$, $v(x_0 + \ell, y) = (x_0 + \ell + \alpha \theta_h(y - y_0), y)$ and $\|Dv - \text{Id}\|_{L^\infty} \leq c\alpha$.

Proof. Under the same assumptions as in Lemma 2.1, and with the same domain subdivision, we define

$$v_1(x, y) = \begin{cases} x + \alpha y & \text{on } \omega_1, \\ x - \alpha y + \alpha \frac{h}{4} + 2\alpha\gamma(x) & \text{on } \omega_2, \\ x + \alpha y - \frac{\alpha h}{2} & \text{on } \omega_3, \\ x - \alpha y + \frac{3\alpha h}{4} - 2\alpha\gamma(x) & \text{on } \omega_4, \\ x + \alpha y - \alpha h & \text{on } \omega_5. \end{cases}$$

In this case the strain terms $\partial_1 v_1$ lie in a diagonal entry of Dv and therefore their energy contribution cannot be reduced using the other component and the rotations, therefore we can simply take $v_2(x, y) = y$. The estimate (2.1) is optimal, and correspondingly we obtain $\text{dist}(Dv, K) \leq c\alpha h/\ell$ everywhere. The strain gradient term can be estimated in the same way as in Lemma 2.1. \square

The boundary layer is also similar.

Lemma 2.5. *Let $0 < h \leq \ell$, $\omega = (x_0, x_0 + \ell) \times (y_0, y_0 + h)$, $\alpha \in (0, 1)$, $\varepsilon > 0$. Then there is a function $v : \omega \rightarrow \mathbb{R}^2$ such that*

$$E_1^\varepsilon[v, \omega] \leq c(\alpha^2 h \ell + \alpha \varepsilon \ell)$$

with the boundary conditions $v(x, y) = (x, y)$ for $y \in \{y_0, y_0 + h\}$ or $x = x_0$, $v(x_0 + \ell, y) = (x_0 + \ell + \alpha \theta_h(y - y_0), y)$ and $\|Dv - \text{Id}\|_{L^\infty} \leq c\alpha$.

Proof. We use the domain subdivision from Lemma 2.2 with

$$v_1(x, y) = \begin{cases} y + \alpha y & \text{on } \omega'_B, \\ y + \frac{1}{4}\alpha h \gamma\left(\frac{x}{\ell}\right) & \text{on } \omega'_M, \\ y + \alpha\left(\frac{h}{2} - y\right) & \text{on } \omega_A, \\ y - \frac{1}{4}\alpha h \gamma\left(\frac{x}{\ell}\right) & \text{on } \omega''_M, \\ y + \alpha(y - h) & \text{on } \omega''_B, \end{cases}$$

and estimate the various energy terms as in Lemma 2.4. \square

We remark that in both Lemma 2.5 and Lemma 2.4 a simpler choice of γ (for example, $\gamma(x) = x$) would have been possible, since no derivatives of γ enter the definition of v .

Lemma 2.6. *Under the assumptions of Theorem 1.1, there are two functions $u, v \in \mathcal{M}$ such that*

$$E_1^\varepsilon[u, \Omega] \leq c \min \left\{ \alpha^{4/3} \varepsilon^{2/3} H L^{1/3} + \alpha \varepsilon L, \alpha^2 L H \right\}$$

and

$$E_1^\varepsilon[v, \Omega] \leq c \min \left\{ \alpha^4 L H + \alpha^{4/3} \varepsilon^{2/3} L H^{1/3} + \alpha \varepsilon H, \alpha^2 L H \right\}.$$

Proof. The construction of u is identical to the one discussed in Lemma 2.3, with Lemma 2.5 instead of Lemma 2.2 and Lemma 2.4 instead of Lemma 2.1. Summing the series we obtain

$$E_1^\varepsilon[u, \Omega'] \leq c \left(\frac{\alpha^2 H^3}{N^2 L} + \alpha \varepsilon N L \right).$$

We choose $N = \lceil \alpha^{1/3} H / (\varepsilon^{1/3} L^{2/3}) + 4H/L \rceil$ and conclude

$$E_1^\varepsilon[u, \Omega] \leq c \alpha^{4/3} \varepsilon^{2/3} H L^{1/3} + c \alpha \varepsilon (L + H).$$

The treatment of the bound $\alpha^2 L H$ and the elimination of the term $\alpha \varepsilon H$ are analogous to the other case.

We now turn to v . In the case K_1 there are two rank-one connections, and if H is much larger than L it is convenient to use a rotated pattern with long thin stripes along e_2 . This construction can be obtained by rotating the one just derived. We set

$$Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and identify it with a linear map from \mathbb{R}^2 to \mathbb{R}^2 . Let u be the function constructed above, using $\tilde{H} = L$, $\tilde{L} = H$. Then u is the identical map on the boundary of $\tilde{\Omega} = (0, H) \times (0, L)$. We set

$$v(x, y) = Zu(Z(x, y)).$$

Then v obeys the required boundary conditions on $\partial\Omega$ and $|D^2v|(\Omega) = |D^2u|(\tilde{\Omega})$. In order to estimate the strain energy, we observe that $Dv = ZDu(Z(x, y))Z$. We assert that for all $F \in \mathbb{R}^{2 \times 2}$ one has

$$\text{dist}(ZFZ, K) \leq \text{dist}(F, K) + \alpha^2. \quad (2.5)$$

This will then imply

$$E_1^\varepsilon[v, \Omega] \leq 2E_1^\varepsilon[u, \tilde{\Omega}] + 2\alpha^4 L H,$$

as required.

It remains to prove (2.5). We first show that there is a rotation $R_a \in \text{SO}(2)$ such that $|R_a Z A_1 Z - A_1| \leq \alpha^2$, and correspondingly for B_1 . We set

$$R_a = \frac{1}{\sqrt{1 + \alpha^2}} \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \in \text{SO}(2)$$

and compute

$$R_a Z A_1 Z = \begin{pmatrix} \sqrt{1 + \alpha^2} & -\alpha/\sqrt{1 + \alpha^2} \\ 0 & 1/\sqrt{1 + \alpha^2} \end{pmatrix}$$

which gives $|R_\alpha Z A_1 Z - A_1| \leq \alpha^2$ for all $\alpha \in (-1, 1)$. Let now $Q \in \text{SO}(2)$, $F \in \mathbb{R}^{2 \times 2}$ be such that $\text{dist}(F, K) = |F - Q A_1|$ (the case with B_1 is the same). We compute

$$\begin{aligned} \text{dist}(ZFZ, K) &\leq |ZFZ - ZQA_1Z| + \text{dist}(ZQA_1Z, K) \\ &= |F - QA_1| + \text{dist}(ZA_1Z, K) \leq \text{dist}(F, K) + \alpha^2, \end{aligned}$$

where we used that $Z \in \text{O}(2) \setminus \text{SO}(2)$ implies $ZQZ \in \text{SO}(2)$, and obtain (2.5). \square

3 Lower bounds

3.1 General Strategy

As in many singularly perturbed nonconvex problems, the lower bound arises by the interaction of the following effects: the second-gradient term limits the oscillations of the gradient in the interior of the domain, and therefore brings the deformation u away from the interpolation of the boundary data. The strain term then makes a deviation of the deformation from the interpolation of the boundary data expensive. The main point in the proof is to make these effects quantitative; in the present case the main difficulty resides in the gradient term, which does not control individual derivatives but only the deviation of the deformation gradient from the set of matrices K_j .

In order to make the strategy quantitative, it is convenient to localize at the appropriate length scale. We shall denote by λ a length scale chosen at the end of the proof, qualitatively it can be understood as a typical length scale of the microstructure in the interior of the domain. Since energy can concentrate, we shall choose (in Lemma 3.1(i)) a square of side λ as well as an horizontal and a vertical stripe of width λ on which the energy is no higher than on average. Then we shall show (Lemma 3.1(ii)) that on this square any low-energy deformation is approximately affine, with gradient in K_j . In order to relate this to the boundary data, exploiting the elastic energy on the stripes, we need to eliminate rotations from the picture. This is done in Lemma 3.3, which shows that if a vector has length approximately unity, and one component averages (because of the boundary data) to 1, then the vector is approximately constant. We then separate the two different settings we consider. In the case K_1 , addressed in Section 3.2, the key remaining difficulty is the treatment of the two possible orientations for the microstructure, which requires a separation of two cases. In the case K_2 , addressed in Section 3.3, the key difficulty is in the treatment of the rotational invariance of the elastic energy and the degeneracy of the rank-one connection. As we know from the upper bound, see discussion in (2.2), the two components can interact to substantially reduce the energy. An optimal

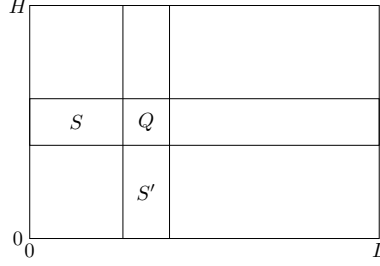


Figure 6: Geometry in Lemma 3.1.

lower bound will therefore require a careful treatment of this interaction. This will be done using a suitable test function to integrate twice by parts, see Lemma 3.6 below. In this section we use $x = (x_1, x_2)$ for an element of \mathbb{R}^2 .

Lemma 3.1 (Localization). *Assume $H, L > 0$, $j \in \{1, 2\}$, $u \in \mathcal{M}$, $\alpha \in (0, 1)$, $\varepsilon > 0$. For any $\lambda \in (0, \min\{L, H\}]$ there are stripes $S = (0, L) \times (s, s + \lambda) \subset \Omega$, $S' = (s', s' + \lambda) \times (0, H) \subset \Omega$ such that, writing for brevity $E_j = E_j^\varepsilon[u, \Omega]$, the following holds:*

(i)

$$E_j^\varepsilon[u, S] \leq c \frac{\lambda}{H} E_j, \quad (3.1)$$

$$E_j^\varepsilon[u, S'] \leq c \frac{\lambda}{L} E_j, \quad (3.2)$$

$$E_j^\varepsilon[u, S \cap S'] \leq c \frac{\lambda^2}{LH} E_j, \quad (3.3)$$

$$\|\partial_1 u_2\|_{L^1(S \cap S')} \leq c \frac{\lambda^2}{LH} \|\partial_1 u_2\|_{L^1(\Omega)}. \quad (3.4)$$

(ii) *There are $F \in K_j$ and $a \in \mathbb{R}^2$ such that*

$$\int_{S \cap S'} |Du - F| dx \leq c \frac{\lambda^3}{\varepsilon LH} E_j + c \frac{\lambda^2}{L^{1/2} H^{1/2}} E_j^{1/2} \quad (3.5)$$

and

$$\int_{S \cap S'} |u(x) - Fx - a| dx \leq c \frac{\lambda^4}{\varepsilon LH} E_j + c \frac{\lambda^3}{L^{1/2} H^{1/2}} E_j^{1/2}. \quad (3.6)$$

The constant c is universal.

Proof. (i): We subdivide Ω into $M = \lfloor H/\lambda \rfloor \geq H/(2\lambda)$ disjoint horizontal stripes $S_k = (0, L) \times (k\lambda, (k+1)\lambda)$, where $\lfloor t \rfloor = \max\{z \in \mathbb{Z} : z \leq t\}$. Since

$$\sum_{k=0}^{M-1} E_j^\varepsilon[u, S_k] \leq E_j$$

we have

$$\#\left\{k \in [0, M) \cap \mathbb{N} : E_j^\varepsilon[u, S_k] \geq \frac{5}{M} E_j\right\} \leq \frac{M}{5}$$

(in particular, for $M < 5$ the set is empty). Analogously we have $N = \lfloor L/\lambda \rfloor \geq L/(2\lambda)$ disjoint vertical stripes $S'_h = (h\lambda, (h+1)\lambda) \times (0, H)$, and at most $N/5$ of them do not satisfy the corresponding bound. Therefore the total number of choices of (k, h) which do not satisfy (3.1) and (3.2), with $c = 10$, is no larger than $2NM/5$.

A similar estimate leads to

$$\#\left\{(k, h) \in [0, M) \times [0, N) \cap \mathbb{N}^2 : E_j^\varepsilon[u, S_k \cap S'_h] \geq \frac{5}{MN} E_j\right\} \leq \frac{MN}{5}.$$

Hence for all but $MN/5$ of the possible choices (3.3) holds, with $c = 20$, and the same for $\|\partial_1 u_2\|_{L^1(S_k \cap S'_h)}$. Since we have four conditions, and each of them excludes at most $\lfloor MN/5 \rfloor$ choices, there are necessarily some left, for which all four conditions are valid. The resulting geometry is illustrated in Figure 6.

(ii): Let F' be the average of Du over $Q = S \cap S'$. Writing $F' = F' - Du + Du$ we obtain

$$\lambda^2 \text{dist}(F', K) \leq \|Du - F'\|_{L^1(Q)} + \|\text{dist}(Du, K)\|_{L^1(Q)}.$$

We define F as the matrix in K closest to F' and write $|Du - F| \leq |Du - F'| + \text{dist}(F', K)$, so that

$$\begin{aligned} \|Du - F\|_{L^1(Q)} &\leq \|Du - F'\|_{L^1(Q)} + \lambda^2 \text{dist}(F', K) \\ &\leq 2\|Du - F'\|_{L^1(Q)} + \|\text{dist}(Du, K)\|_{L^1(Q)}. \end{aligned}$$

Using Poincaré's inequality on the first term and Hölder's inequality on the second, and then recalling (3.3), we obtain

$$\begin{aligned} \|Du - F\|_{L^1(Q)} &\leq c\lambda\|D^2u\|_{L^1(Q)} + \lambda\|\text{dist}(Du, K)\|_{L^2(Q)} \\ &\leq c\frac{\lambda^3}{LH\varepsilon}E_j + c\lambda^2\frac{E_j^{1/2}}{L^{1/2}H^{1/2}} \end{aligned}$$

which concludes the proof of (3.5). To prove (3.6) it suffices to define a as the average of $u(x) - Fx$ over Q and apply Poincaré's inequality once more. \square

The above estimates immediately give the lower bound in the case of very thin domains.

Lemma 3.2 (Thin domains). *Let $L, H > 0$, $\alpha \in (0, 1)$, $\varepsilon > 0$. Then for every $u \in \mathcal{M}$ we have $E_j \geq c \min\{\alpha\varepsilon(L + H), \alpha^2 LH\}$.*

Proof. We use Lemma 3.1 with $\lambda = \min\{L, H\}$ and observe that $Q = S \cap S'$ is a square with at least two sides on $\partial\Omega$. By the trace theorem, (3.5)–(3.6) imply

$$\int_{\partial Q} |u(x) - Fx - a| d\mathcal{H}^1(x) \leq c \frac{\lambda^3}{\varepsilon LH} E_j + c \frac{\lambda^2}{L^{1/2} H^{1/2}} E_j^{1/2}.$$

On the other hand, we show below that there is a universal constant $c_* > 0$ such that

$$c_* \alpha \lambda^2 \leq \int_{\partial Q \cap \partial\Omega} |x - Fx - a| d\mathcal{H}^1(x) \text{ for all } a \in \mathbb{R}^2, F \in K_j. \quad (3.7)$$

Since $u(x) = x$ on $\partial\Omega$ we obtain

$$c_* \alpha \lambda^2 \leq c \frac{\lambda^3}{\varepsilon LH} E_j + c \frac{\lambda^2}{L^{1/2} H^{1/2}} E_j^{1/2}.$$

At least one of the two addends must be larger than $c_* \alpha \lambda^2/2$, therefore

$$E_j \geq c \min\left\{\frac{\alpha \varepsilon LH}{\lambda}, \alpha^2 LH\right\},$$

inserting λ this implies the assertion.

It remains to prove (3.7). After a translation we can assume $Q = (0, \lambda)^2$. Assume first that the two sides are horizontal. The integral on the bottom side gives

$$\int_0^\lambda |te_1 - Fte_1 - a| dt \geq \frac{1}{4} |Fe_1 - 1| \lambda^2$$

since for any $v \in \mathbb{R}^2$ we have

$$\min_{a' \in \mathbb{R}^2} \int_0^\lambda |vt + a'| dt = \frac{1}{4} |v| \lambda^2.$$

Combining the top and bottom sides gives

$$\begin{aligned} \lambda^2 |Fe_2 - e_2| &\leq \int_0^\lambda |(\text{Id} - F)\lambda e_2| dt \\ &\leq \int_0^\lambda |(\text{Id} - F)te_1 - a| + |(\text{Id} - F)(te_1 + \lambda e_2) - a| dt. \end{aligned}$$

Therefore the integral in (3.7) is bounded from below by $|F - \text{Id}| \lambda^2/4$. Swapping x_1 and x_2 , the same holds if the two sides on which we integrate are vertical.

Finally, if $F \in K_2$ we have $|Fe_2| = 1 \pm \alpha$, which implies $|Fe_2 - e_2| \geq \alpha$ and concludes the proof with $c_* = 1/4$.

If $F \in K_1$ then

$$\begin{aligned} \alpha &= |Fe_1 \cdot Fe_2| = |(Fe_1 - e_1) \cdot Fe_2 + e_1 \cdot (Fe_2 - e_2)| \\ &\leq 2|Fe_1 - e_1| + |Fe_2 - e_2| \leq 3|F - \text{Id}|, \end{aligned}$$

and the proof is concluded with $c_* = 1/12$. \square

Before closing this general part we present a lemma which permits to deal with the boundary data.

Lemma 3.3. *Let $\omega \subset \mathbb{R}^n$ be bounded and measurable, $v \in L^1(\omega; \mathbb{R}^2)$, $d \in L^2(\omega; [0, \infty))$ and $e \in S^1$ be such that $v \cdot e - 1$ has average 0 and $|v| \leq 1 + d$ almost everywhere. Then*

$$\|v \cdot e - 1\|_{L^1} \leq 2|\omega|^{1/2}\|d\|_{L^2}$$

and

$$\|v \cdot e^\perp\|_{L^1} \leq 3|\omega|^{3/4}\|d\|_{L^2}^{1/2} + |\omega|^{1/2}\|d\|_{L^2}.$$

All norms are taken over ω .

Proof. We assume $e = e_1$ without loss of generality. Since $v_1 - 1$ has average 0, we have (writing $f_\pm = \max\{\pm f, 0\}$)

$$\begin{aligned} \|v_1 - 1\|_{L^1} &= \int_{\omega} (v_1 - 1)_+ + (v_1 - 1)_- \\ &= 2 \int_{\omega} (v_1 - 1)_+ \leq 2 \int_{\omega} (|v| - 1) dx \leq 2\|d\|_{L^1}. \end{aligned}$$

This proves the first inequality. To address the second one, we write $v_2^2 = |v|^2 - v_1^2$ which implies

$$\|v_2\|_{L^2}^2 = \|v\|_{L^2}^2 - \|v_1\|_{L^2}^2.$$

The last term can be estimated, writing $1 = v_1 + (1 - v_1)$, by

$$|\omega| - 2\|d\|_{L^1} \leq \|v_1\|_{L^1} \leq |\omega|^{1/2}\|v_1\|_{L^2}.$$

If $2\|d\|_{L^1} \leq |\omega|$, squaring and recalling $|v| \leq 1 + d$ we obtain

$$\begin{aligned} \|v_2\|_{L^2}^2 &\leq \|(1 + d)^2\|_{L^1} - (|\omega|^{1/2} - 2|\omega|^{-1/2}\|d\|_{L^1})^2 \\ &= |\omega| + 2\|d\|_{L^1} + \|d\|_{L^2}^2 - |\omega| + 4\|d\|_{L^1} - 4|\omega|^{-1}\|d\|_{L^1}^2 \\ &\leq 6\|d\|_{L^1} + \|d\|_{L^2}^2. \end{aligned}$$

In the case $2\|d\|_{L^1} \geq |\omega|$ we instead write

$$\|v_2\|_{L^2}^2 \leq \|(1 + d)^2\|_{L^1} \leq |\omega| + 2\|d\|_{L^1} + \|d\|_{L^2}^2 \leq 4\|d\|_{L^1} + \|d\|_{L^2}^2.$$

In both cases,

$$\|v_2\|_{L^2}^2 \leq 6\|d\|_{L^1} + \|d\|_{L^2}^2 \leq 6|\omega|^{1/2}\|d\|_{L^2} + \|d\|_{L^2}^2.$$

Another application of Hölder's inequality gives

$$\|v_2\|_{L^1} \leq |\omega|^{1/2}(6|\omega|^{1/2}\|d\|_{L^2} + \|d\|_{L^2}^2)^{1/2} \leq 3|\omega|^{3/4}\|d\|_{L^2}^{1/2} + |\omega|^{1/2}\|d\|_{L^2}.$$

□

3.2 Proof of the lower bound in Theorem 1.1

At this point we specialize to the first case. We write for brevity $E = E_1 = E_1^\varepsilon[u, \Omega]$ and $K = K_1$.

Lemma 3.4. *Assume $H, L > 0$, $\alpha \in (0, 1)$, $\varepsilon > 0$, $u \in \mathcal{M}$. Let S, S' be as in Lemma 3.1. Then*

$$\int_{S \cap S'} |u_1(x) - x_1| dx \leq c\lambda^2 \frac{L^{1/2}}{H^{1/2}} E^{1/2} \quad (3.8)$$

and

$$\int_{S \cap S'} |u_2(x) - x_2| dx \leq c\lambda^2 \frac{H^{1/2}}{L^{1/2}} (E + \alpha^4 LH)^{1/2}. \quad (3.9)$$

Further,

$$\|\partial_1 u_2\|_{L^1(\Omega)} \leq c(HL)^{3/4} E^{1/4} + c(HL)^{1/2} E^{1/2}. \quad (3.10)$$

Proof. This follows from repeated application of Lemma 3.3. We first use $v = \partial_1 u$, $e = e_1$, $d = \text{dist}(Du, K)$, $\omega = S$. By the boundary conditions $\int_S (\partial_1 u_1 - 1) dx = 0$, and $|v| \leq 1 + d$ because $|Fe_1| = 1$ for all $F \in K$ and $|(Du)e_1| \leq \min_{F \in K} |Du - F| + |Fe_1| = 1 + \text{dist}(Du, K)$. We obtain, using (3.1),

$$\|\partial_1 u_1 - 1\|_{L^1(S)} \leq c(\lambda L)^{1/2} \left(\frac{\lambda E}{H} \right)^{1/2} = c\lambda \frac{L^{1/2} E^{1/2}}{H^{1/2}}.$$

Integrating,

$$\|u_1 - x_1\|_{L^1(S \cap S')} \leq \lambda \|\partial_1 u_1 - 1\|_{L^1(S)}$$

proves (3.8). Using $\omega = \Omega$ with the same v , e , and d we get

$$\|\partial_1 u_2\|_{L^1(\Omega)} \leq 3(HL)^{3/4} E^{1/4} + (HL)^{1/2} E^{1/2},$$

which proves (3.10). We now turn to the other component, and set $v = \partial_2 u$, $e = e_2$, $d = \text{dist}(Du, K) + \alpha^2$, $\omega = S'$. Since for all $F \in K$ we have $|Fe_2| = \sqrt{1 + \alpha^2} \leq 1 + \alpha^2/2$ we obtain $|v| = |(Du)e_2| \leq 1 + \frac{1}{2}\alpha^2 + \text{dist}(Du, K)$. In this case, recalling (3.2),

$$\|d\|_{L^2(S')}^2 \leq c \frac{\lambda E}{L} + 2\lambda H \alpha^4 \leq c \frac{\lambda(E + \alpha^4 LH)}{L}.$$

Proceeding as above, Lemma 3.3 gives

$$\|\partial_2 u_2 - 1\|_{L^1(S')} \leq c(\lambda H)^{1/2} \left(\frac{\lambda(E + \alpha^4 LH)}{L} \right)^{1/2}$$

and therefore (3.9). \square

Lemma 3.5. *There is $c > 0$ such that under the assumptions of Theorem 1.1 one has, for any $u \in \mathcal{M}$,*

$$E_1^\varepsilon[u, \Omega] \geq c \min \left\{ \alpha^{4/3} \varepsilon^{2/3} L^{1/3} H + \alpha \varepsilon L, \right. \\ \left. \alpha^{4/3} \varepsilon^{2/3} L H^{1/3} + \alpha \varepsilon H + \alpha^4 L H, \alpha^2 L H \right\}.$$

Proof. We fix $\lambda \in (0, \min\{L, H\}]$, the precise value will be chosen below, and choose the stripes S and S' as in Lemma 3.1.

We first show that there is $c_* > 0$ be such that for every $F \in K$ and $a \in \mathbb{R}^2$ one has

$$\begin{aligned} \text{if } |F_{21}| < \alpha/2 \text{ (case 1): } \quad c_* \alpha \lambda^3 &\leq \|(x - Fx - a) \cdot e_1\|_{L^1(S \cap S')}, \\ \text{if } |F_{21}| \geq \alpha/2 \text{ (case 2): } \quad c_* \alpha \lambda^3 &\leq \|(x - Fx - a) \cdot e_2\|_{L^1(S \cap S')}. \end{aligned}$$

The second one is immediate. To prove the first one we observe that $F \in K$ implies $F_{21} = \sin \theta$, $F_{12} = \pm \alpha \cos \theta - \sin \theta$ for some $\theta \in \mathbb{R}$, hence in the first case we have $|F_{12}| \geq \alpha(\sqrt{3/4} - 1/2) \geq c\alpha$, leading to the required estimate.

For $i \in \{1, 2\}$ we write

$$|(x - Fx - a) \cdot e_i| \leq |(u(x) - x) \cdot e_i| + |u(x) - Fx - a|$$

we obtain in case i

$$c_* \alpha \lambda^3 \leq \|u_i(x) - x_i\|_{L^1(S \cap S')} + \|u(x) - Fx - a\|_{L^1(S \cap S')}. \quad (3.11)$$

At this point we distinguish the two cases.

Case 1 ($|F_{21}| < \alpha/2$). We estimate the first term in (3.11) by (3.8) and the second by (3.6),

$$c_* \alpha \lambda^3 \leq c \lambda^2 \frac{L^{1/2}}{H^{1/2}} E^{1/2} + c \frac{\lambda^4}{\varepsilon L H} E + c \frac{\lambda^3}{L^{1/2} H^{1/2}} E^{1/2}. \quad (3.12)$$

Since $\lambda \leq L$ the last term can be absorbed into the first one, and

$$c_* \alpha \lambda^3 \leq c \lambda^2 \frac{L^{1/2}}{H^{1/2}} E^{1/2} + c \frac{\lambda^4}{\varepsilon L H} E \quad (3.13)$$

which can be rewritten as

$$E \geq c \min \left\{ \frac{\alpha^2 H \lambda^2}{L}, \frac{\alpha \varepsilon L H}{\lambda} \right\}.$$

The right-hand side is maximized by choosing $\lambda = (\varepsilon/\alpha)^{1/3} L^{2/3}$. Since we can only choose $\lambda \in (0, \min\{L, H\}]$, we set

$$\lambda = \min \left\{ (\varepsilon/\alpha)^{1/3} L^{2/3}, L, H \right\}$$

and obtain

$$E \geq c \min \left\{ \alpha^{4/3} \varepsilon^{2/3} L^{1/3} H, \alpha^2 L H, \frac{\alpha^2 H^3}{L} \right\}.$$

We recall that by Lemma 3.2 we have $E \geq c \min \{ \alpha \varepsilon (L + H), \alpha^2 L H \}$. Either $E \geq \alpha^2 L H$ or E is larger than the first expression, therefore we can combine the two estimates to produce

$$E \geq c \min \left\{ \alpha^{4/3} \varepsilon^{2/3} L^{1/3} H + \alpha \varepsilon (L + H), \frac{\alpha^2 H^3}{L} + \alpha \varepsilon (L + H), \alpha^2 L H \right\}.$$

We show that the second entry in the minimum can be dropped. Indeed,

$$\alpha^{4/3} \varepsilon^{2/3} L^{1/3} H = \left(\frac{\alpha^2 H^3}{L} \right)^{1/3} (\alpha \varepsilon L)^{2/3}$$

shows that the second term is larger (up to a factor) than the first one. Analogously, if $\alpha \varepsilon H \geq \alpha^{4/3} \varepsilon^{2/3} L^{1/3} H$ then $\varepsilon \geq \alpha L$, but in this case the minimum equals $\alpha^2 L H$. Therefore we can drop the addend $\alpha \varepsilon H$ in the first term. We conclude that in case 1

$$E \geq c \min \left\{ \alpha^{4/3} \varepsilon^{2/3} L^{1/3} H + \alpha \varepsilon L, \alpha^2 L H \right\}. \quad (3.14)$$

Case 2 ($|F_{21}| \geq \alpha/2$). If $\|\partial_1 u_2\|_{L^1(Q)} \leq \lambda^2 \alpha/4$, then (3.5) gives

$$\frac{1}{4} \lambda^2 \alpha \leq c \frac{\lambda^3}{\varepsilon L H} E + c \frac{\lambda^2}{L^{1/2} H^{1/2}} E^{1/2}$$

which implies (3.12), hence this case has already been treated. Otherwise, from (3.4) and (3.10) we immediately obtain

$$\frac{1}{4} \alpha \lambda^2 \leq c \frac{\lambda^2}{L H} \|\partial_1 u_2\|_{L^1(\Omega)} \leq c \lambda^2 \left(\frac{E}{L H} \right)^{1/4} + c \lambda^2 \left(\frac{E}{L H} \right)^{1/2}.$$

If $E \geq L H$ there is nothing to prove, hence we can ignore the second term. Otherwise

$$E \geq c \alpha^4 L H, \quad (3.15)$$

which in particular permits to estimate

$$E + \alpha^4 L H \leq c E. \quad (3.16)$$

Using (3.9) and (3.6) in (3.11) yields

$$c_* \alpha \lambda^3 \leq c \lambda^2 \frac{H^{1/2}}{L^{1/2}} (E + \alpha^4 L H)^{1/2} + c \frac{\lambda^4}{\varepsilon L H} E + c \frac{\lambda^3}{L^{1/2} H^{1/2}} E^{1/2}.$$

The last term can be absorbed into the first one, using (3.16) we obtain

$$c_* \alpha \lambda^3 \leq c \lambda^2 \frac{H^{1/2}}{L^{1/2}} E^{1/2} + c \frac{\lambda^4}{\varepsilon L H} E$$

for all $\lambda \in (0, \min\{L, H\}]$. This is the same as (3.13) up to swapping L and H , and gives

$$E \geq c \min \left\{ \alpha^{4/3} \varepsilon^{2/3} H^{1/3} L, \frac{\alpha^2 L^3}{H}, \alpha^2 L H \right\}.$$

Recalling Lemma 3.2,

$$E \geq c \min \left\{ \alpha^{4/3} \varepsilon^{2/3} H^{1/3} L + \alpha \varepsilon (L + H), \frac{\alpha^2 L^3}{H} + \alpha \varepsilon (L + H), \alpha^2 L H \right\}$$

and we drop the same irrelevant terms as in the previous case. Recalling (3.15) we conclude

$$E \geq c \min \left\{ \alpha^{4/3} \varepsilon^{2/3} H^{1/3} L + \alpha \varepsilon H + \alpha^4 L H, \alpha^2 L H \right\},$$

which together with (3.14) concludes the proof. \square

3.3 Proof of the lower bound in Theorem 1.2

We now come to the proof of the lower bound in Theorem 1.2. In this case the local structure leads to oscillations in $\partial_2 u_2$, and the branching can only be horizontal.

There are two ways of producing a lower bound on the energy this. The first one is to estimate $\partial_1 u_2$ using Lemma 3.3, as was done in [CC14]. This gives the optimal $\varepsilon^{4/5}$ scaling, but not the optimal scaling in the parameter α . Indeed, this argument is based on a purely nonlinear effect, and would not produce any bound in the linear setting.

We show here that the bound from [CC14] can be improved. The key idea is to use *two* partial integrations, to pass from the $\partial_2 u_2$ term to the $\partial_1 u_1$ one, and to use for the leading-order term the estimate for $\partial_1 u_1$ in Lemma 3.3, which scales as E instead of $E^{1/2}$. Swapping the indices between the partial derivative and the component of u is done using the invariance under rotations. In order to be able to do the repeated partial integration it is helpful to test the derivatives with a smooth test function, instead of simply integrating over a suitable domain as done in Section 3.2.

The key result in this section, which proves the lower bound of Theorem 1.2, is the following.

Lemma 3.6. *There is $c > 0$ such that under the assumptions of Theorem 1.2 one has, for any $u \in \mathcal{M}$,*

$$E_2^\varepsilon[u, \Omega] \geq c \min \left\{ \alpha^{6/5} \varepsilon^{4/5} L^{1/5} H + \alpha \varepsilon L, \alpha^2 L H \right\}.$$

Proof. We fix $\lambda \in (0, \min\{L, H\}]$, the precise value will be chosen below. We choose the stripes S and S' with Lemma 3.1 and set $Q = S \cap S'$, $I = (s, s + \lambda)$.

Let F be as in Lemma 3.1(ii). We choose x_1^* such that $I^* = \{x_1^*\} \times I \subset Q$ and, using (3.5),

$$\int_I |Du - F|(x_1^*, t) dt \leq \frac{1}{\lambda} \int_Q |Du - F| dx \leq c \frac{\lambda^2}{\varepsilon LH} E + c \frac{\lambda E^{1/2}}{L^{1/2} H^{1/2}}. \quad (3.17)$$

We exploit the boundary conditions, using again Lemma 3.3. In particular, we set $v = \partial_1 u$, $e = e_1$, $d = \text{dist}(Du, K)$, $\omega = S$ and obtain, recalling (3.1),

$$\|\partial_1 u_1 - 1\|_{L^1(S)} \leq c\lambda \frac{L^{1/2} E^{1/2}}{H^{1/2}} \quad (3.18)$$

and

$$\|\partial_1 u_2\|_{L^1(S)} \leq c\lambda \frac{L^{3/4} E^{1/4}}{H^{1/4}} + c\lambda \frac{L^{1/2} E^{1/2}}{H^{1/2}}.$$

Since in the case $E \geq HL$ there is nothing to prove, we can replace the last estimate with the simpler one

$$\|\partial_1 u_2\|_{L^1(S)} \leq c\lambda \frac{L^{3/4} E^{1/4}}{H^{1/4}}. \quad (3.19)$$

This will only be used in the estimate of the error term arising from the linearization of $\text{SO}(2)$, in the second line of (3.20). The leading-order terms will be estimated with (3.18).

At this point the proof starts to differ significantly from the one for K_1 . We fix one test function $\varphi \in C_c^\infty([0, 1]; [0, 1])$ with $\varphi = 1$ on $(1/4, 3/4)$, and define $f \in C_c^\infty(I)$ by $f(\lambda t + s) = \varphi(t)$, so that

$$\|f\|_{L^1(I)} \geq \frac{\lambda}{2}, \quad \|f\|_{L^\infty} = 1, \quad \|f'\|_{L^\infty} \leq \frac{c}{\lambda}, \quad \|f''\|_{L^\infty} \leq \frac{c}{\lambda^2}.$$

We distinguish two cases, depending on the value of F_{22} .

Case 1: $|F_{22} - 1| \geq \alpha/2$. With a triangular inequality and a partial integration we write

$$\begin{aligned} \frac{\alpha}{4}\lambda &\leq \frac{\alpha}{2}\|f\|_{L^1(I)} \leq \left| \int_I (F_{22} - 1)f dx_2 \right| \\ &\leq \left| \int_I (F_{22} - \partial_2 u_2)f dx_2 \right| + \left| \int_I (\partial_2 u_2 - 1)f dx_2 \right| \\ &\leq \int_I |Du - F| dx_2 + \left| \int_I (u_2 - x_2)f'(x_2) dx_2 \right| \\ &\leq \int_I |Du - F| dx_2 + \left| \int_{(0, x_1^*) \times I} \partial_1 u_2(x) f'(x_2) dx \right| \quad (3.20) \end{aligned}$$

where we used $\|f\|_{L^\infty} = 1$ and in the last step the boundary data $u_2(0, x_2) = x_2$. In all integrals over I the functions u and Du are evaluated at (x_1^*, x_2) .

At this point we trade $\partial_1 u_2$ with $\partial_2 u_1$ in the last term. To do this we use the structure of rotations. First we choose $G \in L^\infty(S; K)$ with $|Du - G| = \text{dist}(Du, K)$ almost everywhere, then choose $\sigma \in L^\infty(S; \{-1, 1\})$ such that $G \in \text{SO}(2)\text{diag}(1, 1 + \sigma\alpha)$ almost everywhere. A direct computation shows that $G_{12} + (1 + \sigma\alpha)G_{21} = 0$ and therefore

$$(1 + \sigma\alpha)\partial_1 u_2 + \partial_2 u_1 = (1 + \sigma\alpha)(\partial_1 u_2 - G_{21}) + (\partial_2 u_1 - G_{12}).$$

We multiply this expression with $f'(x_2)$ and integrate over $(0, x_1^*) \times I$ to obtain

$$\begin{aligned} \left| \int_{(0, x_1^*) \times I} \partial_1 u_2(x) f'(x_2) dx \right| &\leq \left| \int_{(0, x_1^*) \times I} \partial_2 u_1(x) f'(x_2) dx \right| \\ &\quad + \alpha \|f'\|_{L^\infty} \int_{(0, x_1^*) \times I} |\partial_1 u_2(x)| dx \\ &\quad + 3 \int_S |f'(x_2)| |Du - G|(x) dx. \end{aligned} \quad (3.21)$$

We treat the different terms separately. The term with $\partial_2 u_1$ is integrated by parts and then estimated using (3.18),

$$\begin{aligned} \left| \int_{(0, x_1^*) \times I} \partial_2 u_1(x) f'(x_2) dx \right| &= \left| \int_{(0, x_1^*) \times I} (u_1(x) - x_1) f''(x_2) dx \right| \\ &\leq \|f''\|_{L^\infty} \|u_1 - x_1\|_{L^1(S)} \\ &\leq L \|f''\|_{L^\infty} \|\partial_1 u_1 - 1\|_{L^1(S)} \\ &\leq L \frac{c}{\lambda^2} c \lambda \frac{L^{1/2} E^{1/2}}{H^{1/2}} = \frac{c}{\lambda} \frac{L^{3/2} E^{1/2}}{H^{1/2}}. \end{aligned}$$

For the second term we use $\|f'\|_{L^\infty} \leq c/\lambda$ and (3.19) to obtain

$$\alpha \|f'\|_{L^\infty} \int_{(0, x_1^*) \times I} |\partial_1 u_2|(s, x_2) dx \leq c \alpha \frac{L^{3/4} E^{1/4}}{H^{1/4}}.$$

Finally, the last term is bounded using (3.1),

$$\int_S |f'| |Du - G| dx \leq \|f'\|_{L^\infty} \int_S \text{dist}(Du, K) dx \leq c \frac{L^{1/2} E^{1/2}}{H^{1/2}}.$$

The estimate (3.20) becomes, using (3.17), (3.21) and the three previous estimates

$$\alpha \lambda \leq c \frac{\lambda^2}{\varepsilon L H} E + c \frac{\lambda E^{1/2}}{L^{1/2} H^{1/2}} + \frac{c}{\lambda} \frac{L^{3/2} E^{1/2}}{H^{1/2}} + c \alpha \frac{L^{3/4} E^{1/4}}{H^{1/4}} + c \frac{L^{1/2} E^{1/2}}{H^{1/2}}. \quad (3.22)$$

Since $\lambda \leq L$ the second term and the last one can be absorbed into the third one, and we have

$$\alpha \lambda \leq c \frac{\lambda^2}{\varepsilon L H} E + \frac{c}{\lambda} \frac{L^{3/2} E^{1/2}}{H^{1/2}} + c \alpha \frac{L^{3/4} E^{1/4}}{H^{1/4}}$$

for all $\lambda \in (0, \min\{L, H\}]$. Therefore

$$E \geq c \min\left\{\frac{\alpha\varepsilon LH}{\lambda}, \frac{\alpha^2\lambda^4 H}{L^3}, \frac{\lambda^4 H}{L^3}\right\}.$$

Since $\alpha \leq 1$ the last term can be ignored. Finally, we choose λ as

$$\lambda = \min\left\{\frac{\varepsilon^{1/5} L^{4/5}}{\alpha^{1/5}}, L, H\right\}$$

and obtain

$$E \geq c \min\left\{\alpha^{6/5} \varepsilon^{4/5} L^{1/5} H, \alpha^2 LH, \frac{\alpha^2 H^5}{L^3}\right\}.$$

By Lemma 3.2 we have $E \geq c \min\{\alpha\varepsilon(L+H), \alpha^2 LH\}$. Therefore

$$E \geq c \min\left\{\alpha^{6/5} \varepsilon^{4/5} L^{1/5} H + \alpha\varepsilon(L+H), \alpha^2 LH, \frac{\alpha^2 H^5}{L^3} + \alpha\varepsilon(L+H)\right\}.$$

Since $(\frac{\alpha^2 H^5}{L^3})^{1/5} (\alpha\varepsilon L)^{4/5} = \varepsilon^{4/5} \alpha^{6/5} L^{1/5} H$, the last term is always larger than the first one and can be dropped. Further, if $\varepsilon\alpha H \geq \alpha^{6/5} \varepsilon^{4/5} L^{1/5} H$ then $\varepsilon \geq \alpha L$, and therefore the relevant energy bound is $\alpha^2 LH$. We conclude

$$E \geq c \min\left\{\alpha^{6/5} \varepsilon^{4/5} L^{1/5} H + \alpha\varepsilon L, \alpha^2 LH\right\}$$

and therefore the proof in this case.

Case 2: $|F_{22} - 1| < \alpha/2$. Since $|Fe_2| = 1 \pm \alpha$, in this case necessarily $|F_{12}| \geq \alpha/2$. Therefore, proceeding as above,

$$\begin{aligned} \frac{1}{4}\alpha\lambda &\leq \frac{1}{2}\alpha\|f\|_{L^1(I)} = \left|\int_I F_{12}f(x_2)dx_2\right| \\ &\leq \left|\int_I (F_{12} - \partial_2 u_1)f(x_2)dx_2\right| + \left|\int_I \partial_2 u_1 f(x_2)dx_2\right| \\ &\leq \int_I |Du - F|dx_2 + \left|\int_I (u_1 - x_1)f'(x_2)dx_2\right|. \end{aligned}$$

In turn, recalling (3.18),

$$\left|\int_I (u_1 - x_1)f'(x_2)dx_2\right| \leq \|f'\|_{L^\infty} \int_{(0, x_1^*) \times I} |\partial_1 u_1 - 1| dx \leq c \frac{L^{1/2} E^{1/2}}{H^{1/2}}.$$

Using (3.17) we obtain also in this case (3.22), which concludes the proof. \square

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